Estimation of Lorentz Force From Dimensional Analysis:
Similarity Solutions and Scaling Laws

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Abstract—In this paper, we consider the use of dimensional analysis for modeling electromagnetic levitation and braking problems, which are described by the Lorentz force law. Based on Maxwell's equations, to illustrate the underlying field problem, we formulate a complete mathematical model of a simple academic example, where a permanent magnet is moving over an infinite plate at constant velocity. The step-by-step procedure employed for dimensional analysis is described in detail for the given problem. A dimensionless model with a reduced number of parameters is obtained, which highlights the dominant dependences, and it is invariant to the dimensional system employed. Using the dimensionless model, a concise parametric study is conducted to illustrate the advantages of the dimensionless representation for displaying complex data in an efficient manner. We provide an exhaustive study of the dependences of the Lorentz force on the dimensionless parameters to complete the analysis, and we give results for a generalized representation of the problem. Finally, scaling laws are derived and illustrated based on practical examples.

Index Terms—Dimensional analysis, eddy currents, Lorentz force, magnetic levitation, permanent magnet (PM), scaling laws, similarity.

I. INTRODUCTION

When an electrically conducting object moves through a magnetic field, such as that produced by a permanent magnet (PM) or a current-carrying coil, motion-induced eddy currents appear inside the conductor, which lead to electromagnetic levitation and braking forces. The phenomenon is well described by the Lorentz force law, and the corresponding technical developments have been made in this area for over 100 years.

At present, this phenomenon had numerous different applications, including magnetic bearing [1]–[3], coupling [4], precision actuation [5], [6], magnetic suspension [7]–[11], and energy harvesters [12], [13]. However, the best known technical applications are magnetically levitated trains, which use electromagnets, PMs, or superconducting magnets for levitation and guidance. Magnetically levitated trains were first proposed by Bachelet [14] in 1912, but high-speed transportation systems became popular in the early 1970s and they have evolved into a recognized form of modern transportation in the 21st century. There has been great success in the development of sufficient expressions of the underlying field problem [15]–[22]. During the 1990s, great efforts were made [23]–[28], which laid the foundations for the current world speed record in a test run of 603 km/h [29].

More recently, the Lorentz force phenomenon has been extensively investigated in the context of non-destructive testing and in the evaluation of electrically conductive materials using the PMs. Two different approaches are employed for material characterization to use the secondary magnetic field obtained from the motion-induced eddy currents: measuring the secondary field using magnetic field probes [30], [31] or measuring the Lorentz force acting on the PM [32]–[34].

In educational applications, the slowing down of a magnet falling in a non-ferromagnetic, electrically conducting pipe is employed as a popular demonstration to introduce engineering and physical science students to the basics of electromagnetic induction phenomena. This problem has been extensively studied by focusing on experimental, analytical, or numerical solutions [35]–[43].

Notable contributions to the falling magnet problem using dimensional analysis were made in [44]–[46], where these studies briefly demonstrated the possibility of estimating the terminal velocity of the magnet based on the dimensional analysis supported by laboratory experiments.

The phenomenon itself is well understood and fully described by the magnetic field transport equation [47], which can describe many different problems, and it is an integral part of engineering design for numerous devices with electromagnetic interactions. However, even seemingly simple tasks are actually difficult to solve and they require modern simulation software, as well as specially trained professionals in the field of electromagnetics. In most cases, the dependences of the different variables are not obvious and their complex interactions make it difficult to gain familiarity with the phenomenon. Another difficulty is caused by the high number of different parameters in the applications of electromagnetic levitation and braking. Thus, many optimization tasks are extremely time-consuming and often uneconomical.

In this paper, we contribute to the solution of technical levitation problems by considering the idea of Lorentz force estimation with the help of dimensional analysis for a rather simple academic problem to illustrate the principal steps, thereby aiding the understanding of how to handle more complex applications. In Section II, we briefly describe a...
The equations with the appropriate boundary conditions can be analytically solved using the 2-D Fourier transform approach [17], [20], [49].

The Lorentz force exerted on the PM is calculated using Parseval’s theorem [50] as

\[
F = \frac{\mu_0}{4\pi^2} \Re e \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{H}^* \cdot \hat{D} \, dk_x \, dk_y \, dz \right)
\]

where \(k_x, k_y\) are the transform variables and \(\hat{H}^*\) is the Fourier transform of the magnetic field associated with the eddy currents induced in the conductor. The 2-D Fourier transform \(\hat{J}_s\) of the source current density is given by

\[
\hat{J}_s = [\hat{J}_{sx}, \hat{J}_{sy}]^T = j \pi J_0 D J_1 \left( \frac{D}{2} \right) \left[ \frac{k_y}{k} - \frac{k_x}{k} \right]^T
\]

where \(k^2 = k_x^2 + k_y^2\). \(J_0 = B_i/\mu_0\) is the magnitude of the source current density, and \(J_1(\cdot)\) denotes the first-order Bessel function of the first kind. The components of the force \(F\) are given by the formulas

\[
F_x = \frac{\mu_0}{2\pi^2} \int_0^\infty \int_0^\infty m |T(k, \beta)| (k_x |\hat{J}_{sy}|^2 - k_y \hat{J}_{sy} \hat{J}_{sx}) \times \frac{(1 - e^{-kH_2})^2}{k^3} e^{-2kh} \, dk_x \, dk_y
\]

(6)

\[
F_y = 0
\]

(7)

\[
F_z = \frac{\mu_0}{2\pi^2} \int_0^\infty \int_0^\infty \Re e |T(k, \beta)| (|\hat{J}_{sx}|^2 + |\hat{J}_{sy}|^2) \times \frac{(1 - e^{-kH_2})^2}{k^2} e^{-2kh} \, dk_x \, dk_y
\]

(8)

with \(T(k, \beta)\) obtained as

\[
T(k, \beta) = \frac{(\beta^2 - 1) \tanh \beta kt}{2\beta(1 + \beta^2) \tanh \beta kt}
\]

(9)

where \(\beta = \alpha/k\) and \(\alpha^2 = j \mu_0 \sigma v k_t + k^2\).

The set of equations (6)–(8) builds the mathematical model for the problem under investigation by describing the relation between the dependent parameter of interest \(F\) and eight relevant physical parameters \(v, \mu_0, \sigma, B_i, D, H, h, \) and \(t\).

We use the presented approach to provide reliable data for the subsequent analysis. Clearly, other methods, such as finite-element analysis or laboratory experiments, would also be suitable for obtaining valid solutions for the force, but the latter would require an additional effort in terms of uncertainty analysis.

### III. Dimensional Analysis

In this section, we present the main contribution of this paper. We apply the dimensional analysis procedure to the defined problem using the mathematical reformulation given by Price [51]. We assume that any complete physical relation must be dimensionally consistent, which is also known as the statement of dimensional homogeneity [52]–[54]. Furthermore, we acknowledge that any physical relationship that is expressed by a complete equation must be invariant to the applied dimensional system [53]–[55].

![Fig. 1. Geometry and parameters of the problem under investigation. A vast, non-magnetic, electrically conducting plate moves rectilinear with a constant velocity under the axially magnetized cylindrical PM at rest.](image-url)
A. Definition of the Physical Model

The first step in the dimensional analysis procedure is the preliminary physical analysis of the system and the definition of the problem, which is described in detail in Section II. The next step is to create a list of the physical parameters \( x_i \) of \( x = \{x_1, x_2, \ldots, x_I\} \), which are expected to be relevant to the features of the phenomena of interest. These parameters should be described using a consistent system of units \([G] = \{[G_1], [G_2], \ldots, [G_K]\}\), which comprise fundamental units \([G_k]\) that are sufficient to define the magnitude of any physical quantity [56]. It should be mentioned that it is customary (as suggested by Maxwell) to denote the dimensions of a quantity \( \phi \) by \([\phi] \) [57]. The dimension \([x_i]\) of any physical parameter \( x_i \) can be written as the product of the powers of the fundamental units

\[
[x_i] = \prod_{k=1}^{K} [G_k]^{d_{ki}} \tag{10}
\]

where \( d_{ki} \) equals to the power to which the \( k \)th fundamental unit is raised in the \( i \)th physical parameter of \( x \). To improve the clarity of the description, the dimensional analysis employs the International System of Units (SI), but the reader is free to choose any other appropriate system [58].

We start the list with the first important parameter in this problem, which is the force \( F \) acting on the PM. We want to perform the dimensional analysis in the scalar form, so the force as a vector quantity has to be decomposed into orthogonal components. Due to the symmetry of the problem, only \( F_x \) and \( F_z \) are of interest because \( F_y \equiv 0 \). Additional parameters comprise the magnitude of the relative velocity \( v \), the electrical conductivity of the plate \( \sigma \), and the PM’s remanence \( B_r \), which are assumed to be relevant to the acting force. The next parameter in the list is the magnetic permeability \( \mu = \mu_0 \mu_r \), where \( \mu_0 \) is the vacuum permeability and \( \mu_r \) is the relative permeability of the plate. We want to restrict the investigation to non-ferromagnetic materials (\( \mu_r \equiv 1 \)), so we only have to consider the vacuum permeability \( \mu_0 \) in our list. It should be mentioned that the vacuum permeability \( \mu_0 \) appears in our list, because we describe all the parameters in SI units. Other systems of units would also lead to other constants, such as the electromagnetic velocity in Gaussian and Heaviside-Lorentz systems. Finally, we employ a group of geometrical parameters to describe all the lengths and distances in our problem, i.e., the cylinder’s diameter \( D \) and height \( H \), the distance between the PM and plate \( h \), and the plate’s thickness \( t \).

The full list contains ten \((I = 10)\) parameters

\[
\mathbf{x} = \{F_x, F_z, v, \sigma, B_r, \mu_0, D, H, h, t\} \tag{11}
\]

which comprise the physical model \( \mathbf{x} \) of our problem. The result of the second step is summarized in Table I, where the dimensions are in fundamental units.

![Table I: List of the Physical Parameters and Constants](image)

Clearly, all the elements of the physical model \( \mathbf{x} \) can be described using a reduced base of \( K = 4 \) fundamental units expressed in terms of mass \([G_1] = M\), length \([G_2] = L\), time \([G_3] = T\), and electric current \([G_4] = I\).

A comprehensive form to represent all the elements of \( \mathbf{x} \) and their corresponding dimensions is the dimensional matrix \( \mathbf{D} \), where the elements \( d_{ki} \) are given in (10) such that

\[
\mathbf{D} = \begin{bmatrix}
1 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -3 & 0 & 1 & 1 & 1 & 1 \\
-2 & -2 & -1 & 3 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & -2 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

for example, the dimension of the velocity \( v \) is the product of the fundamental unit \([L]\) raised to the power of \( 1 \) and \([T]\) to the power of \(-1\).

To continue the analysis, we must assert that the physical model is complete, i.e., it includes all the parameters required to build a correct mathematical model. This model must be dimensionally homogeneous and, thus, invariant to the dimensional system used. This is evident from the previous problem definition (Section II), but in general, the completeness would only be a hypothesis without any knowledge of the mathematical model.

B. Calculation of a Dimensionless Basis Set

Using the complete physical model, we can define a functional relationship that includes all the previously identified parameters. Without loss of generality, this relationship can be written as

\[
g(\mathbf{x}) = g(F_x, F_z, v, \sigma, B_r, \mu_0, D, H, h, t) = 0 \tag{13}
\]

where \( g \) is an unknown function. From Buckingham’s \( \Pi \)-theorem [55], we know that a dimensionally homogeneous equation can be reduced to a relation of independent dimensionless parameters \( \Pi_j \) for a basis set \( \Pi = \{\Pi_1, \ldots, \Pi_I\} \), such that

\[
G(\Pi) = G(\Pi_1, \Pi_2, \ldots, \Pi_J) = 0 \tag{14}
\]

where \( G \) still is an unknown function, but \( J \leq I \). Each dimensionless parameter \( \Pi_j \) is a product of the powers of the governing parameters \( x_i \) with independent dimensions [57]

\[
\Pi_j = \prod_{i=1}^{I} x_i^{s_{ij}}, \quad j = 1 \ldots J \tag{15}
\]
where \( s_{ij} \) denotes the power to which the \( i \)th physical parameter is raised in the \( j \)th dimensionless element of \( \Pi \).

We are interested in finding this basis set, so we consider the dimensional equations of this statement

\[
\left[ \Pi_j \right] = \prod_{i=1}^{I} \left[ x_i \right]^{s_{ij}}.
\] (16)

Analogous to (10), we describe the dimension of each element \( \Pi_j \) as a product of the powers of the fundamental units

\[
\left[ \Pi_j \right] = \prod_{k=1}^{K} \left[ G_k \right]^{c_{kj}}.
\] (17)

If they are dimensionless, each \( \left[ \Pi_j \right] \) must be equal to one, so we can conclude that \( c_{kj} = 0, \forall k, j \). Furthermore, we write the dimensional formula for the right-hand side of (16) using (10), such that

\[
\prod_{i=1}^{I} \left[ x_i \right]^{s_{ij}} = \prod_{i=1}^{I} \left( \prod_{k=1}^{K} \left[ G_k \right]^{d_{ki}} \right)^{s_{ij}}.
\] (18)

By combining (18) and (17), the dimensional equation (16) can be written as

\[
\prod_{k=1}^{K} \left[ G_k \right]^{c_{kj}} = \prod_{i=1}^{I} \left( \prod_{k=1}^{K} \left[ G_k \right]^{d_{ki}} \right)^{s_{ij}}.
\] (19)

For the sake of simplicity, we rewrite (19) by taking the logarithm of both the sides as

\[
\sum_{k=1}^{K} c_{kj} \log[G_k] = \sum_{i=1}^{I} s_{ij} \sum_{k=1}^{K} d_{ki} \log[G_k] \quad \forall j
\] (20)

which holds in the non-trivial case for \( [G_k] \neq 1 \) only if

\[
c_{kj} = \sum_{i=1}^{I} s_{ij} d_{ki} \quad \forall j, k.
\] (21)

Using the matrix notation, it is evident that the unknown basis set of dimensionless parameters is equal to the non-trivial solutions of the homogeneous system of linear equations

\[
c_j = D s_j, \quad c_j \equiv 0 \quad \forall j
\] (22)

with the dimensional matrix \( D \) given by (12) and the basis set vectors \( s_j \). \( D \) is underdetermined, so infinitely many solutions form a vector space. The vector space dimension is equal to \( J \), the number of dimensionless products in a complete set of parameters. This so-called nullity of \( D \) is stated by the rank-nullity theorem of linear algebra

\[
J = \text{nul}(D) = I - \text{rk}(D)
\] (23)

where \( I \) is the number of columns and \( \text{rk} \) is the rank of the dimensional matrix \( D \). Consequently, from the physical model (11) with \( I = 10 \) parameters and the dimensional matrix (12) of rank \( \text{rk}(D) = 4 \), the problem is fully described by a set of \( J = 6 \) independent dimensionless products \( \Pi_j \).

The solutions set of (22) represents the kernel (null space) of \( D \), which can be calculated using the Gaussian elimination. Given that \( D \) is initially built using an arbitrary ordering of the physical parameters, and then the row echelon form of the underdetermined system mainly depends on the arrangement selected. Therefore, we are free to reorder the columns of the dimensional matrix in any form desired. In the following, we use a slightly modified matrix \( D \), where the physical parameter \( h \) is moved to the fourth place of \( x \), to obtain a sparse null space basis. This yields a clearly arranged result and the distance \( h \) is set to the characteristic length of the problem.

The calculation of a rational null space of this reordered matrix yields six vectors that correspond to dimensionless parameters \( \Pi_j \), which are combined into the solution matrix \( S = [s_1; s_2; \ldots; s_J] \) as follows:

\[
S = \begin{bmatrix}
F_x & -1 & 1/2 & 0 & 0 & 0 & 0 \\
F_z & 1 & 0 & 0 & 0 & 0 & 0 \\
\nu & 0 & 1/2 & 1 & 0 & 0 & 0 \\
h & 0 & 3/2 & 1 & -1 & -1 & 1 \\
\sigma & 0 & 1/2 & 1 & 0 & 0 & 0 \\
\oslash & 0 & 0 & 0 & 1 & 0 & 0 \\
D & 0 & 0 & 0 & 1 & 0 & 0 \\
H & 0 & 0 & 0 & 0 & 1 & 0 \\
t & 0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}.
\] (24)

The corresponding set of dimensionless parameters \( \Pi_j \) is constructed using (15) for each solution vector \( s_j \), where \( x^j \) is employed as a shorthand notation for this computation, as suggested in [51]. The calculated set of independent dimensionless parameters \( \Pi \)

\[
\Pi_1 = x^{s_1} = F_z/F_x
\] (25a)

\[
\Pi_2 = x^{s_2} = \sqrt{\nu B_0 h^3 / F_x}
\] (25b)

\[
\Pi_3 = x^{s_3} = \mu_0 \sigma h
\] (25c)

\[
\Pi_4 = x^{s_4} = D/h
\] (25d)

\[
\Pi_5 = x^{s_5} = H/h
\] (25e)

\[
\Pi_6 = x^{s_6} = t/h
\] (25f)

comprises a dimensionless model of our problem, and this is the result of the third step. In contrast to the previously defined physical model \( x \), the new formulation is invariant to the dimensional system used. More importantly, it fully describes the phenomenon of interest using only six parameters instead of the original ten parameters.

\[C. Discussion and Reformulation of the Dimensionless Basis\]

The last step of the dimensional analysis is to evaluate and interpret the derived dimensionless basis set in the light of observations or confirmed mathematical models. To be able to interpret these results it is useful to discuss (25) in the given form.

The first dimensionless parameter \( \Pi_1 \) is the ratio of both the force components, \( F_z \) and \( F_x \), which can be intuitively interpreted as the direction of the force in the \( xz \) plane. The second parameter \( \Pi_2 \) illustrates the remarkable features of a
dimensional analysis by indicating the dominant dependences of the parameters. This expression clearly agrees with the statements in [59] about the Lorentz force acting on a magnetic dipole located at a distance $L$ above a semi-infinite electrically conducting fluid. An estimate is given by the proportionality $F \propto \mu_0^2 \sigma v m L^{-5}$, where $m$ is the magnetic dipole moment $m \propto B_r L^3/\mu_0$. The third dimensionless parameter $\Pi_3$ is called the magnetic Reynolds number $R_m$, which is well known in magnetohydrodynamics, where it indicates the ratio of magnetic advection relative to magnetic diffusion [47].

The last three dimensionless parameters indicate the shape or geometric similarity [52], and they describe the relative sizes of the bodies involved.

This particular set of dimensionless parameters might be too abstract for a convenient description of the Lorentz force exerted on the PM in our problem. Therefore, we are interested in a form that clearly separates independent and dependent variables, as well as their parameters. The basis set vectors $s_j$ are orthogonal and they span the null space, so any linear combination of these vectors is a solution of the homogeneous system (22). Thus, we are free to transform the initial basis set by multiplying the dimensionless parameters with each other to any desired power.

In the reformulated basis set, we define the independent variable $\Pi_1$ as the magnetic Reynolds number $R_m$. The variables $\tilde{F}_x$ and $\tilde{F}_z$, where tilde indicates a dimensionless force component, are the linear combinations of (25a)–(25c). Furthermore, we define the dimensionless geometric parameters $\delta$, $\zeta$, and $\tau$ by rearranging (25d)–(25f).

After some simple calculations and reordering, we obtain a reformulated basis set

$$
R_m := \Pi_1 = \mu_0 \sigma v h \quad (26a)
$$

$$
\delta := \Pi_2 = D/h \quad (26b)
$$

$$
\zeta := \Pi_3 = D/H \quad (26c)
$$

$$
\tau := \Pi_4 = t/h \quad (26d)
$$

$$
\tilde{F}_x := \Pi_5 = \mu_0 F_x / (B_r h) \quad (26e)
$$

$$
\tilde{F}_z := \Pi_6 = \mu_0 F_z / (B_r h) \quad (26f)
$$

which provides a more suitable representation for the following discussion.

### IV. RESULTS AND DISCUSSION

#### A. Dimensionless Representation of Complex Data

In this section, we present the calculated force components that act on the PM to evaluate the derived dimensionless basis set.

The procedure of the force evaluation implemented in MATLAB is shown schematically in Fig. 2. To demonstrate the advantages of the dimensionless representation, we discuss the concise parametric study depicted in Table II for four different settings. We consider the PMs of various sizes with typical magnetic remanences for neodymium–iron–boron (NdFeB) PMs. The electrical conductivity of each plate is in the range for aluminum and copper alloys. All the settings define the systems that are similar in shape to each other, and thus they have identical dimensionless parameters $\delta$, $\zeta$, and $\tau$.

#### Table II

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
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<tbody>
<tr>
<td>$B_r$</td>
<td>T</td>
<td>1.4</td>
<td>1.1</td>
<td>1.2</td>
<td>1.3</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>MS/m</td>
<td>25</td>
<td>45</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$h$</td>
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<td>0.75</td>
<td>1.00</td>
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</tr>
<tr>
<td>$D$</td>
<td>mm</td>
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<td>15.00</td>
<td>20.00</td>
<td>25.00</td>
</tr>
<tr>
<td>$H$</td>
<td>mm</td>
<td>7.50</td>
<td>11.25</td>
<td>15.00</td>
<td>18.75</td>
</tr>
<tr>
<td>$t$</td>
<td>mm</td>
<td>25.00</td>
<td>37.50</td>
<td>50.00</td>
<td>62.50</td>
</tr>
</tbody>
</table>

The geometric similarity is a necessary condition for the complete similarity in this paper.

The results of the parametric study $S_1$–$S_4$ are shown in Fig. 3. The dimensional representations of the calculated force components on the PM are shown in Fig. 3(a) versus the velocity up to $v = 35$ m/s. As expected, all four settings differ in terms of the magnitude of the forces generated. In addition, the characteristic points $P_1$–$P_4$ of intersection for the force components differ in terms of their velocity and force. Nevertheless, it is clear that all the four settings share a common characteristic shape for the resulting forces. This is clearer when we look at the results for the dimensionless representation shown in Fig. 3(b).

As predicted, all the four configurations yield dimensionless representations with identical results in terms of both their magnitude and shape. The resulting force components $F_x$ and $F_z$ merge in the dimensionless representation at identical magnetic Reynolds numbers on two separate curves, and thus they share one point of intersection $P_0$. Clearly, identical magnetic Reynolds numbers do not imply that the velocities are the same, but they do indicate electromagnetic similarity. This is why it is not possible to replace the abscissa in Fig. 3(b) by a dimensional representation of the velocity without defining the remaining parameters of $R_m$. Furthermore, it is clear that the magnet’s remanence $B_r$ does not affect the electromagnetic similarity, and it merely comprises a scaling factor of the second power for the generated force components.

#### B. Dependence on $R_m$

In the following study, we illustrate that a dimensionless representation remains valid without further constraints and it can be used to highlight the dominant dependences in the problem under consideration. For this reason, we extend the
that are much smaller than the speed of light. Therefore, we restrict our investigation to $R_m \leq 10^3$, which is expected to give reliable results.

Fig. 4 shows the force components calculated for fixed $\delta$, $\xi$, and $\tau$ depending on the magnetic Reynolds number $R_m$. It is clear that for both the Lorentz force components, two regions exist where the problem is dominantly described by a particular power of $R_m$. We can clearly distinguish one region with low and another with high $R_m$. In the region where $R_m \leq 10^{-1}$, the dimensionless force components $\tilde{F}_x$ and $\tilde{F}_z$ are proportional to $R_m$ and $R_m^2$, respectively. In the region where $R_m \geq 10^1$, $\tilde{F}_x$ is proportional to $R_m^{-1/2}$, whereas $\tilde{F}_z$ goes to saturation. This observation confirms our hypothesis regarding the dependence on $R_m$.

We recall that the magnetic Reynolds number $R_m$ is a measure of the relative strength of advection and diffusion, and thus their respective characteristics can be attributed directly to the corresponding phenomena. Between the regions of low and high $R_m$ (moderate $R_m$), both the phenomena occur side by side, thereby preventing further characterization using power-law dependences.

C. Dependence on $\tau$

Next, we investigate how the dimensionless Lorentz force components depend on the dimensionless geometric parameters $\delta$, $\xi$, and $\tau$. Therefore, we expand our previous study based on the considerations of the influence of the plate thickness $\tau$ on the force components. This analysis should have a significant impact, because $\tau$ determines the available region where the eddy currents are induced. For this purpose, we express a similar power-law hypothesis for the dependence on $\tau$.

Fig. 5 shows the dependences of the Lorentz force components on the dimensionless plate thickness $\tau$, for fixed $R_m$, $\delta$, and $\xi$. It is clear that for both the force components, two regions exist where the problem is dominantly described by a particular power of $\tau$. We can clearly distinguish one region with low and another with high $\tau$. In the region where $\tau \leq 10^{-1}$, $\tilde{F}_x \propto \tau$ and $\tilde{F}_z \propto \tau^2$, whereas in the region where $\tau \geq 10^2$, $\tilde{F}_x$ and $\tilde{F}_z$ have no further dependence on $\tau$. This observation also confirms our power-law hypothesis about the dependence on $\tau$.

In the following, we refer to the region that has a constant power-law dependence on $\tau$ with thin plate behavior and the region without dependence on $\tau$ with infinite half-space behavior. Between these two regions, where a moderate $\tau$ dominates, no further characterization is useful without the additional investigation of the explicit eddy current density distribution.

The next step of our investigation is to analyze how the dependences of the force components on the dimensionless plate thickness $\tau$ change for different values of the magnetic Reynolds number $R_m$. Fig. 6 shows filled contour plots for the common logarithm of the force components as functions of the magnetic Reynolds number $R_m$ and dimensionless plate thickness $\tau$ for $\delta = 10$ and $\xi = 1$. The force component generated is constant along each contour line. Each value is depicted as
Fig. 4. Dimensionless Lorentz force components \( \tilde{F}_x \) (blue curve) and \( \tilde{F}_z \) (magenta curve) as functions of the magnetic Reynolds number \( R_m \) for fixed \( \delta, \xi, \) and \( \tau \). Blue region: transition zone for the mixed dominance of advection and diffusion, which separates low and high \( R_m \).

Fig. 5. Dimensionless Lorentz force components \( \tilde{F}_x \) (blue curve) and \( \tilde{F}_z \) (magenta curve) as functions of the dimensionless plate thickness \( \tau \) for fixed \( R_m, \delta, \) and \( \xi \). Blue region: transition zone that separates small and large \( \tau \).

the power to base 10 in the color bar. The contour interval employed, i.e., the difference in elevation between successive contour lines, is constant in each graph. Thus, the distance between two lines is a measure of the gradient for a force component at a certain point, which is always perpendicular to the contour lines. At a dimensionless thickness \( \tau \geq 10^1 \), we again observe two regions where both the force components are proportional to the powers of \( R_m \), similar to that given in Fig. 4, but they are invariant to \( \tau \). In the region where \( \tau \leq 10^{-1} \), \( \tilde{F}_x \) is proportional to \( \tau \) and \( \tilde{F}_z \) to \( \tau^2 \) until a characteristic Reynolds number, which is proportional to \( \tau \).

In addition, the maximum force \( \tilde{F}_x \) exceeds that for large dimensionless thickness. After the force maximum is reached, \( \tilde{F}_x \) is inversely proportional to \( R_m \) and \( \tau \), whereas \( \tilde{F}_z \) again goes to saturation, and thus it has no further dependence on either \( R_m \) or \( \tau \). This observation also confirms our hypothesis regarding the power-law dependence on \( \tau \).

D. Generalized Representation

Furthermore, we are interested in generalizing these statements of proportionality. We have seen that it is possible to
Fig. 6. (a) Dimensionless drag force \( \tilde{F}_x \) and (b) dimensionless lift force \( \tilde{F}_z \) as functions of the magnetic Reynolds number \( R_m \) and the dimensionless plate thickness \( \tau \) for fixed \( \delta = 10 \) and \( \xi = 1 \).

distinguish between the regions that are dominantly described by either diffusion (low \( R_m \)) or advection (high \( R_m \)), and the regions with and without dependences on the plate thickness, so we can define characteristic values \( R_{mC} \) and \( \tau_{C} \) that approximately differ between each of these regions. We define \( R_{mC,x,z} \) as the value of the maximum curvature of \( \log(\tilde{F}_{x,z}) \) for \( \tau = 10^3 \) (approximation of infinite half-space) and with fixed \( \delta \) and \( \xi \). Analogously, we define \( \tau_{C,x,z} \) for \( R_m = 10^{-3} \) (low \( R_m \)). As a result, we obtain the estimates of the equilibriums for the different phenomena. To avoid the need for multiple partial derivations to calculate the maximum curvature, we use a simple geometric estimate. The basic idea is to transform the dimensionless force components obtained into an almost symmetric representation

\[
\tilde{F}_{\text{sym}}^x = \frac{1}{\tau^{1/2} R_m^{1/2}} \\
\tilde{F}_{\text{sym}}^z = \frac{1}{\tau R_m} \\
\tilde{F}_{\text{sym}}^z = \frac{1}{\tau R_m}
\] (27)

shown in Fig. 7.

Using this transformation, the characteristic values \( R_{mC,x,z} \) and \( \tau_{C,x,z} \) are defined as

\[
R_{mC,x,z}(\delta, \xi) = \arg \max_{\tau_0 = 10^3, R_m \in \mathbb{R}} \tilde{F}_{\text{sym}}^{x,z}(R_m, \tau_0, \delta, \xi) \\
\tau_{C,x,z}(\delta, \xi) = \arg \max_{R_m=10^{-3}, \tau \in \mathbb{R}} \tilde{F}_{\text{sym}}^{x,z}(R_m, \tau, \delta, \xi)
\] (28)

whereas \( R_{mC,x,z} \) and \( \tau_{C,x,z} \) are the only functions of the dimensionless geometric parameters \( \delta \) and \( \xi \). The term symmetric helps to clarify that the absolute values of the exponents of proportionality in the separated regions are equal to each other after transformation. This also ensures that only one characteristic point exists for all the possible sets of parameters. Equations (28a) and (28b) are numerically evaluated using a derivative-free minimization of the negative symmetric representation \( -\tilde{F}_{\text{sym}}^{x,z} \) defined by (27). The minimization is based on golden section search and parabolic interpolation provided by the MATLAB function \texttt{fminbnd}[60].

Fig. 8 shows the determination of the characteristic values for an arbitrary set of \( \delta \) and \( \xi \). This description of characteristic points is equally valid for our generalization, such as that of the maximum curvature. However, it has particular advantages in the case of real measurements where multiple derivations would lead to incorrect results due to the amplified sensor noise.

When these two characteristic values are determined for a specific set of \( \delta \) and \( \xi \), we can normalize \( \tau \) and \( R_m \) from Fig. 6. Furthermore, we can normalize the dimensionless force components against a characteristic value of interest, e.g., the maximum force in a specific parameter range or the force at one of the two characteristic points of the maximum curvature.
Fig. 8. Determination of the characteristic values (a) $RmCx,z$ along $\tau = 10^3$ and (b) $z_{Cx,z}$ along $Rm = 10^{-3}$ after the transformation of the dimensionless force components $\tilde{F}_{x,z}$ into a symmetric representation $\tilde{F}_{sym}$.

Figures 9 and 10 show filled contour plots for the common logarithm of the force components normalized to their maxima as the functions of the generalized magnetic Reynolds number $Rm/RmCx,z$ and the generalized dimensionless plate thickness $\tau/\tau_{Cx,z}$ for arbitrary $\delta$ and $\xi$. The normalized force component generated is constant along each contour line. Each value is depicted as the power to base 10 in the color bar, and the order of magnitude is given relative to the maximum force in the range considered. The contour intervals employed are again constant in each graph.

In Fig. 9, the parameter space for the normalized drag force $\tilde{F}_x/\tilde{F}_{x,max}$ is divided into four regions. The red curve separates an infinite half-space and thin-plate behavior. The blue curve distinguishes dominantly diffusive (low $Rm$) and advective (high $Rm$) regions. Each region has specific proportionality to the powers of $Rm$ and $\tau$. The slope of the red curve for the values of $Rm/Rm_{C,x} \geq 1$ can be obtained using the proportionality in the two regions with high $Rm$ as

$$\frac{\tau}{\tau_{C,x}} = \left( \frac{Rm}{Rm_{C,x}} \right)^{-1/2}, \quad \frac{Rm}{Rm_{C,x}} \geq 1. \quad (29)$$

The slope of the blue curve for the values of $\tau/\tau_{C,x} \leq 1$ can be obtained in a similar manner using the two regions with thin-plate behavior as

$$\frac{Rm}{Rm_{C,x}} = \left( \frac{\tau}{\tau_{C,x}} \right)^{-1}, \quad \frac{\tau}{\tau_{C,x}} \leq 1. \quad (30)$$

The normalized lift force $\tilde{F}_z/\tilde{F}_{z,max}$ (Fig. 10). In contrast to Fig. 9, the parameter space is divided into only three regions in
this case. The blue curve separates dominantly diffusive and advective behavior with the same estimate as that given by (30) but with squared proportionality compared with the normalized force component \( \tilde{F}_{z} / \tilde{F}_{z, \text{max}} \). The distinction between an infinite half-space and thin-plate behavior is only valid for \( R_m / R_m C_z \leq 1 \), whereas for \( R_m / R_m C_z \geq 1 \), \( \tilde{F}_{z} / \tilde{F}_{z, \text{max}} \) goes to saturation and is almost invariant to the changes in \( R_m \) and \( \tau \) (dashed line).

During extensive parametric studies, we observed that the representations, as shown in Figs. 9 and 10, are completely invariant to the changes in the remaining geometrical parameters \( \delta \) and \( \zeta \). Thus, the selected representation includes all the similarity solutions for the defined problem, which are independent of the input parameters selected in \( x \).

The characteristic variables \( R_m C_{x,z} \) and \( \tau C_{x,z} \), as well as the value \( \tilde{F}_{x,z} \) at the specific parameter point, must be known to denormalize the standard dimensionless representations. Furthermore, this type of representation is less suitable for calculating the actual values of the force components, but it helps us to better understand the phenomenon itself.

Furthermore, during our investigations, we observed that the statement of invariance remained valid for the PMs with different base areas, e.g., quadratic or regular octagonal (not explicitly shown here). This observation is rather surprising and it leads to the hypothesis that electromagnetic similarity also exists between the PMs with different geometries. However, this does not necessarily mean that different PMs will induce an identical eddy current distribution inside the conductor or that the dimensionless force components will be the same, but it does imply the existence of identical normalized representations for PMs with different shapes.

Clearly, the specific values of the characteristic variables \( R_m C_{x,z} \) and \( \tau C_{x,z} \) depend on the base area and the geometric parameters of the PM. The dependences for the cylindrical PM are shown in Fig. 11.

It can be seen that the shapes of the curves are similar to \( R_m C_{x,z} \) and \( \tau C_{x,z} \) for both the force components \( \tilde{F}_{x,z} \), as well as for the absolute value of the force \( \tilde{F} \) (not shown). In particular, \( R_m C_{x,z} \) and \( \tau C_{x,z} \) appear to be inversely proportional to each other over the whole range of \( \delta \) and \( \zeta \). This is confirmed by calculating the product of the factors \( R_m C_{x,z} \) and \( \tau C_{x,z} \), and the related standard deviation \( \sigma_D \) (Table III) for the parameter range investigated in Fig. 11.

The mean relative standard deviations \( \sigma_D \) of \( \approx 2.5% \) are probably the results of truncation errors during the numerical integration required to calculate the force components.

The constancy of the products can be used to reduce the effort required to determine \( R_m C \) and \( \tau C \) by calculating one from the other, which simplifies their subsequent application to model experiments. Furthermore, this supports our hypothesis that electromagnetic similarity also exists between the PMs with different geometries.

V. Scaling Laws

We applied dimensional analysis to a simple problem that could be solved by an exact analytical formulation. However, many engineering problems are so complex that no analytical solution can be obtained. In many of these problems, model experiments are the only way to avoid expensive and time-consuming experiments with wide variation in the governing parameters.

We must stress that in the current problem, we are interested in the influence of the parameters \( u \), \( \sigma \), and \( B_{r} \) and the geometric parameters \( D \), \( H \), and \( t \) on the force components \( F_x \) and \( F_z \) that act on the PM. Based on the dimensional analysis, we know the form in which all the parameters must appear in the unknown functions that determine the acting force. From the discussion in Section IV, we know that we can distinguish between different regions of dependences from \( R_m \) and \( \tau \) for the force components. These regions are separated by the areas of transition, which include the defined characteristic points \( R_m C \) and \( \tau C \). These characteristic values are the only functions of the dimensionless diameter of the PM \( \delta \) and the aspect ratio \( \zeta \).
In order to clearly formulate the scaling laws for our problem, we distinguish between a prototype, which is the object of interest, and a model, which we employ to perform experiments under controlled conditions. For the prototype and the model, we can consider three different cases.

In the first case, we perform experiments based on a model with electrodynamic similarity. Electrodynamic similarity includes geometric similarity and it occurs if and only if each dimensionless parameter \( \frac{R_m}{R_{m \text{sym}}} \), \( \frac{\delta}{\delta_{\text{sym}}} \), \( \xi \), and \( \tau \) has the same value in the model and the prototype. When we design the model experiment, we must consider that not only all the geometric parameters need to be scaled linearly. The magnetic Reynolds number \( R_m \) itself also changes with the geometric scale and it must be adapted by changing the product of \( \sigma \). For example, if we use the same material for the plate in an \( n \)-time larger model, then the relative velocity between the plate and the PM must be decreased by \( 1/n \) for \( R_m \) to be constant. The forces obtained from the model experiments should then be rescaled using \( (\tau)^2 \) and \( \tau \) to obtain a correct evaluation. Using the \( n \)-time larger model, we know that the measured forces are \( n^2 \) times larger than those for the prototype. Furthermore, we know that if we use a PM with an \( m \)-time higher magnetic remanence \( B_r \), then the measured forces are also \( m^2 \) times larger. Again, it is clear that we do not necessarily have to use the same grade of PM material to obtain a similar electrodynamic model. We only need to consider these differences when the experimental results are evaluated. All these statements about scaling in the case of electrodynamic similarity are a direct consequence of the dimensional analysis, and an additional discussion of dependences in different regions is not required.

The second and more general case occurs when we allow \( R_m \) and \( \tau \) in the model to differ within a certain range, but where \( \delta \) and \( \xi \) are equal in the model and the prototype. This is achieved by identifying which of the different regions contains the prototype and ensuring that the model experiment occurs in the same region. If the model is closer to the transition zones than the prototype, but it is still outside, then the results can be scaled to those of the prototype. To identify the region of the model, we need to slightly vary \( R_m \) and \( \tau \), and then observe the changes in the measured force components. If the change fits the proportionality of one of the characteristic regions, then the current region is identified and the scaling parameters are known.

In the third case, we allow \( R_m \) and \( \tau \) to differ over the whole range in the model, where the phenomenon is still mainly described by the same physical effects. It is necessary to perform four steps to ensure a correct estimation of the Lorentz force. First, either \( R_m \text{C} / \tau \text{C} \) must be found as described in (28) by varying the free parameters \( \tau \) or \( R_m \), respectively. Second, the other characteristic value must be calculated using the corresponding factor from Table III. Third, the axes in Fig. 9 or 10 should be denormalized by multiplying the axis scale with the respective characteristic value. Finally, the complete graph is denormalized using the dimensionless force component measured by the model at a single arbitrary point. In the result, for a specific configuration of \( \delta \) and \( \xi \), we can estimate the forces for a very large range of settings, but without the need for direct exploration.

As an example of the third case, we take a specific set of \( \delta \) and \( \xi \), i.e., a fixed distance for a specific cylindrical PM. In the first step, the plate thickness \( t \) is changed gradually at a velocity that corresponds to a magnetic Reynolds number \( R_m = 10^3 \). The measured values of the force in the \( x \)-direction \( F_{xn} \) are multiplied by the associated \( t_{n}^{1/2} \) to obtain the symmetric representation given by (27). The characteristic value \( \tau \text{C} \) is estimated at the maximum of \( \dot{F}_{\text{sym}} \) and stored with the measured force \( F_{\text{C}} \). Next, \( R_m \text{C} \) is estimated using Table III for a circular base shape as \( R_m \text{C} = 1.94 \times \tau \text{C} \). Using these three values, Fig. 9 can be denormalized.

VI. CONCLUSION

In this paper, we contributed to the process of modeling and scaling in the Lorentz force applications using dimensional analysis. For this particular problem, we defined a physical model, a list of relevant parameters \( x \), and their individual dimensions \( [x] \). Using this list, we set up a dimensional matrix \( D \) to calculate a dimensionless basis set \( \Pi \) comprising a dimensionless model of the same problem with a reduced number of parameters independent of the dimensional system used. We transformed this basis set to obtain a representation that is easy to handle. We conducted a concise parametric study to illustrate the advantages of the dimensionless representation for displaying complex data in an efficient manner.

In particular, we showed the influence of the magnetic Reynolds number \( R_m \) and the dimensionless plate thickness \( \tau \) for one arbitrary pair of remaining dimensionless geometric parameters. Using a power-law hypothesis for both dependences, we defined four readily distinguished regions, where each can be described by a simple power law.

The positions of the transition zones between separated regions greatly depend on the geometric parameters \( \delta \) and \( \xi \). Therefore, the results were normalized against the characteristic values \( R_m \text{C} \) and \( \tau \text{C} \), which are defined as the points with the maximum curvature of the dimensionless force components \( \dot{F}_{x} \) and \( \dot{F}_{z} \). This normalization yields a generalized representation of the dimensionless force components, which is completely invariant to the changes in the geometric parameters \( \delta \) and \( \xi \). The apparent inversely proportional relationship between the characteristic parameters \( R_m \text{C} \) and \( \tau \text{C} \) for different PM shapes was a surprising result, which was shown to be an additional simplification that facilitates the subsequent formulation of scaling laws.

Finally, we discussed scaling laws for three relevant scenarios, which were illustrated with practical examples.

In conclusion, we must stress that the success of dimensional analysis depends on the absolute requirement that the physical model is complete. This is an unattainable goal for many technical problems, so a satisfactory compromise is required to focus on the relevant aspects of the phenomena included. This compromise depends on the judgments that only come with experience and a continual validation based on observations and numerical implementations of the developed mathematical models.
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